# Critical and supercritical dynamics of quasiperiodic systems

Jukka A. Ketoja
*Department of Mathematics, University of Helsinki, P. O. Box 4, FIN-00014 Helsinki, Finland*

Indubala I. Satija[1]
*Department of Physics and Institute of Computational Sciences and Informatics, George Mason University, Fairfax, VA 22030, USA*

The almost periodic eigenvalue problem described by the Harper equation is connected to other classes of quasiperiodic behaviour: the dissipative dynamics on critical invariant tori and quasiperiodically driven maps. Firstly, the strong coupling limit of the supercritical Harper equation and the strong dissipation limit of the critical standard map play equivalent role in the renormalization analysis of the self-similar fluctuations of localized eigenfunctions and the universal slope of the projected map on the invariant circle. Secondly, we use a simple transformation to relate the Harper equation to a quasiperiodically forced one-dimensional map. In this case, the localized eigenstates of the supercritical Harper equation correspond to strange but nonchaotic attractors of the driven map. Furthermore, the existence of localization in the eigenvalue problem is associated with the appearance of homoclinic points in the corresponding map.

## I. INTRODUCTION

Systems with two competing periodicities have been in the forefront of research in nonlinear dynamics as well as in condensed matter physics. Perhaps the most well-known paradigm in the study of quasiperiodic dynamics in autonomous dissipative and Hamiltonian systems is the standard map with and without dissipation [2,3]. A related problem in the condensed matter physics is the Schrödinger equation with quasiperiodic potential [4]. A simple discrete prototype in this field is the Harper equation [5] or the almost Mathieu equation [4]. Invariant circles in the standard map correspond to extended states of the Harper equation while the breakup of an invariant circle is analogous to the localization transition in the Harper equation. Both transitions have been studied by the renormalization methods [2,6–8].

We have applied the decimation methodology, developed earlier to show the existence of "slope" universality in the dissipative, critical standard map [9], to a variety of eigenvalue problems which can be written in the linear-difference form [10]. Very recently, the renormalization analysis was extended to the Harper equation beyond criticality [11]. These studies have shown the existence of a unique strong coupling fixed point which describes the fluctuations of localized eigenstates in the whole of supercritical region. In the first part of this paper, we show that this strong coupling fixed point of the Harper equation is related to the strong dissipation limit of the critical standard map. The renormalization analysis for the critical standard map is rewritten in the form which makes the correspondence clear.

The second part of this paper is to relate the results of the supercritical Harper equation to the dynamics of quasiperiodically forced dissipative maps. In parallel to the study of quasiperiodic dynamics in autonomous dissipative and Hamiltonian systems there have been investigations on the dissipative dynamics of quasiperiodically forced systems. Although the presence of such forcing causes some usual orbits like the periodic ones to be absent in these systems, there are other interesting features which have not been found in autonomous systems. Perhaps the most striking phenomenon is the appearance of strange nonchaotic attractors (SNA) [12]. These are geometrically strange attracting sets for which all the Lyapunov exponents (except the one related to the forcing) are negative. Inspite of a great deal of research in systems exhibiting SNA [13,14], there are still many open questions: the condition for the existence of SNA as well as a general renormalization theory on their appearance are not fully understood.

Bondeson et al. [15] pointed out an interesting connection between a quasiperiodically forced oscillator and the Schrödinger equation with quasiperiodic potential. Using the substitution introduced by Prüfer [16] already seventy years ago, the two classes of systems can be mapped onto one another the energy eigenvalue of the Schrödinger equation appearing as a parameter for the oscillator. The transition from an invariant circle to SNA in the classical problem is related to the transition from extended to localized states in the quantum problem.

In this paper, we map the discrete Schrödinger equation, namely the Harper equation, to a quasiperiodically driven one-dimensional map so that the eigenfunction corresponds to an attractor of the map. The simplicity of the discrete system compared to the continuum model offers the possibility of obtaining further insight into the existence and characterization of SNA. By focusing on the strong coupling limit, we are able to show the existence of SNA in quasiperiodically driven dissipative maps. Furthermore, this approach also helps in understanding some of the open questions related to the significance of the critical phase in the Harper equation. [8,11] Our focus is somewhat similar to the one of Bondeson et al. [15] in that we are interested in various tran-

sitions and their effects on the renormalization dynamics or the dynamics of the forced map.

In Section II we briefly review the renormalization approach as applied to the Harper equation and the dissipative standard map. In Section III, we show how these two problems are related. In Section IV, we discuss the dynamics of the one-dimensional map associated with quasiperiodic eigenvalue problems and for the first time show explicitly the connection between localization and strange nonchaotic attractors conjectured earlier by Bondeson et al. [15]. In Section V, we discuss our results from a general perspective. As a comparison with previous two quasiperiodic systems, we describe the results with completely different properties for the phonon equation of the Frenkel-Kontorova model although this eigenvalue problem is formally very close to the Harper equation.

## II. RENORMALIZATION APPROACH TO QUASIPERIODIC SYSTEMS: THE HARPER EQUATION AND THE DISSIPATIVE STANDARD MAP

We have developed a decimation approach to study quasiperiodic systems with two incommensurate frequencies which can be written in the following linear-difference form: [9–11]

$$a(i)\psi_{i+1} + b(i)\psi_{i-1} + c(i)\psi_i = 0. \tag{1}$$

Here, $a$, $b$, and $c$ are real or complex functions of $i\sigma$ where $\sigma$ is an irrational number equal to the ratio of two frequencies. Although the decimation method can be implemented for any irrational $\sigma$, here we consider the simplest case where $\sigma$ is given by the inverse golden ratio $\sigma = (\sqrt{5}-1)/2$. Our formulation can be generalized also to the case where $\psi_i$ is a multi-component vector and $a$, $b$, and $c$ are matrices.

In the decimation scheme, it is appropriate to decimate out all sites except those labelled by the Fibonacci numbers $F_n$ (which are the best rational approximants of the golden ratio). At the $n^{th}$ decimation level ($n = 2, 3, ...$), the linear-difference equation is expressed in the form

$$f_n(i)\psi(i + F_{n+1}) = \psi(i + F_n) + e_n(i)\psi(i). \tag{2}$$

The additive property $F_{n+1} = F_n + F_{n-1}$ of the Fibonacci numbers provides exact recursion relations for the decimation functions $e_n$ and $f_n$:

$$e_{n+1}(i) = -\frac{Ae_n(i)}{1 + Af_n(i)} \tag{3}$$

$$f_{n+1}(i) = \frac{f_{n-1}(i + F_n)f_n(i + F_n)}{1 + Af_n(i)} \tag{4}$$

$$A = e_{n-1}(i + F_n) + f_{n-1}(i + F_n)e_n(i + F_n).$$

The decimation functions $e_n$ and $f_n$ define a renormalization flow which converges asymptotically on an attractor. In the case when the attractor is a limit cycle of period $p$, it reflects the translational invariance of the function $\psi$ in the Fibonacci space and also captures the self-similarity underlying it at all scales. The decimation functions $e_n$ and $f_n$ determine the universal scaling ratios

$$\zeta_j = \lim_{n\to\infty} |\psi(F_{pn+j})/\psi(0)|; \;\; j = 0, ..., p-1. \tag{5}$$

The two quasiperiodic systems that we discuss here are the Harper equation

$$\psi_{i+1} + \psi_{i-1} + 2\lambda\cos[2\pi(i\sigma + \phi)]\psi_i = E\psi_i \tag{6}$$

and the dissipative standard map

$$x_{i+1} = x_i + \Omega + b(x_i - x_{i-1}) - \frac{k}{2\pi}\sin(2\pi x_i). \tag{7}$$

In the Harper equation, $E$ is the eigenvalue corresponding to the eigenfunction $\psi_i$. For irrational $\sigma$, the equation describes a particle in a periodic potential incommensurate with the periodicity of the lattice. On the other hand, the standard map with $0 \le b < 1$, and for certain values of $\Omega$, describes quasiperiodic dynamics on an invariant circle. This equation can be written in the linear-difference form by considering the equation for the derivatives $\xi_i = \partial x_i/\partial x_0$ describing the slope of the $i$ times iterated reduced circle map $x_{k+1} = h(x_k)$ at $x_0$:

$$\xi_{i+1} + b\xi_{i-1} - [1 + b - k\cos(2\pi x_i)]\xi_i = 0. \tag{8}$$

In ref. [9], a somewhat similar formulation of the decimation equation (2) and the recursion relations was used to derive the following result along the critical line $k = k_c(b, \Omega)$ for the breakup of the golden invariant circle in the standard map: $\xi_{F_n} b^{-F_n} \to \zeta_S$ as $n \to \infty$, where $\zeta_S = 0.435625...$ and the slope was calculated at the point $x_0$ which lead to the smallest value. In fact, the value of $\zeta$ was calculated in the limit $b \to 0$ and its universality, termed as the "slope universality", for finite $b$ was not understood although suggested by the numerical results. This result, which has not been yet rigorously proven, would have an interesting mathematical implication: under renormalization, the critical renormalization fixed point would attract not only circle maps with zero slope but also those with finite slope. The latter would correspond to critical reduced circle maps of higher dimensional systems with finite Jacobians. They would give a new dimension for the stable manifold of the critical renormalization fixed point.

Later [10], the same decimation procedure was applied to the critical Harper equation with $\lambda = 1$ for fixed $E$ which was usually chosen either at the band center or at the band edges. Here the $p$ universal numbers $\zeta$ described

the amplitude of the wave function at points which were spaced $p$ (or any multiple of $p$) Fibonacci sites apart from the central peak. More recently [11], the modified decimation procedure explained in above was used to describe universal features in the Harper equation *beyond* criticality. [11] Here, the fluctuations of localized eigenfunctions were shown to be universal.

In the next section, we discuss both problems using this modified decimation approach and show that the *supercritical* Harper equation with $\lambda > 1$ and the *critical* dissipative standard map with $b < 1$ are closely related. In particular, this analysis clarifies the universality of $\zeta_S$ for the full range of the dissipation parameter $b$.

## III. ANALOGY BETWEEN THE SUPERCRITICAL HARPER EQUATION AND THE CRITICAL DISSIPATIVE STANDARD MAP

In this section, we explain the common principle behind the slope universality of the critical dissipative standard map [9] and the supercritical universality of the Harper equation [11]. In both cases, the universality can be generalized also to other than the Fibonacci sites but for the present purpose it suffices to consider this restricted universality.

The Harper equation in the supercritical regime ($\lambda > 1$) is characterized by the positive inverse localization length $\gamma = ln(\lambda)$ for the wave function $\psi_i$ [17]. The fluctuations of the localized eigenfunction are obtained by factorizing the exponentially decaying part out. The wave function $\psi_i$ is written as

$$\psi_i = e^{-\gamma i}\eta_i. \tag{9}$$

The resulting equation for the fluctuations is given by the linear-difference form

$$\frac{1}{\lambda}\eta_{i+1} + \lambda\eta_{i-1} + 2\lambda\cos[2\pi(i\sigma+\phi)]\eta_i = E\eta_i. \tag{10}$$

A correspondent equation for the standard map is obtained by factorizing out the exponentially decaying effective Jacobian from the slope. In other words, with $\xi_i = b^i\eta_i$ we obtain

$$b\eta_{i+1} + \eta_{i-1} - [1 + b - k\cos(2\pi x_i)]\eta_i = 0. \tag{11}$$

It is interesting to note that the above equations have well-defined limits when $\lambda \to \infty$ or $b \to 0$, respectively. Moreover, these limits resemble one another. As the latter corresponds to the circle map, for which the existence of a renormalization fixed point has been known for a long time [6,7], the analogy suggests the existence of a similar renormalization fixed point also for the fluctuations in the Harper equation above criticality. In fact, applying the decimation tools to Eq. (10), the fluctuations of states with eigenvalue either at the band center or at the band edges were found to be self-similar characterized by periodic attractors of the renormalization flow [11]. In particular, for the states at the band edges with $\phi = 1/4$, $|\eta_{F_n}/\eta_0| \to \zeta_H = 0.172586410945...$ as $n \to \infty$, independently of the value of $\lambda > 1$. This universality can also be interpreted in an alternative way: the finite-size inverse localization length $\bar{\gamma}$, defined by $\psi_i = e^{-\bar{\gamma} i}$, varies around the asymptotic value $\gamma$ in a universal way given by the equation

$$\gamma - \bar{\gamma} = \log|\eta_{F_n}|/F_n. \tag{12}$$

This is a rather intriguing result as the inverse localization length does depend upon $\lambda$.

In order to understand the origin of the parameter-independent universality in the Harper equation and the dissipative standard map, we write the equation for $\eta_i$ in both cases in the form

$$f_2(i)\eta(i+2) = \eta(i+1) + e_2(i)\eta(i) \tag{13}$$

which gives the first step of a decimation approach of generating equations of the form

$$f_n(i)\eta(i+F_{n+1}) = \eta(i+F_n) + e_n(i)\eta(i). \tag{14}$$

The power of this method lies in the fact that in iterating the recursion relations the decimation function $f_n$ asymptotically "renormalizes" to zero. This is a numerical observation but also suggested by the appearance of the constant $C = 1/\lambda < 1$ (for the Harper equation) or $C = b < 1$ (for the standard map) in front of the function $f_2$. Because one can take the trivial conditions $f_1 \equiv 1$ and $e_1 \equiv 0$, in $f_n$ there appears a multiplying constant $C^{F_{n-1}}$ which tends to zero as $n \to \infty$. In other words, asymptotically the recursion relation for the decimation function $e_n$ simplifies into

$$e_{n+1}(i) = -e_{n-1}(i+F_n)e_n(i) \tag{15}$$

which is the form in the strong coupling limit $\lambda \to \infty$ of the Harper equation or the infinite dissipation limit $b \to 0$ of the standard map. We can therefore expect the renormalization fixed points in these limits to describe the scaling properties of $\eta(i)$ for the whole $\lambda$ or $b$ range. One should also note that because $f_n$ vanishes asymptotically, the scaling ratio $\zeta$ is given simply by the asymptotic value of $e_n$ at the lattice site $i = 0$.

In order to solve for the fixed points, the discrete lattice index $i$ has to be replaced by a continuous variable. For the standard map, we can take this to be simply the original variable $x$ so that the recursion relation becomes

$$e_{n+1}(x) = -e_{n-1}(h^{F_n}(x))e_n(x). \tag{16}$$

This equation appeared also in the previous work [9]. However, because there a variant of the decimation method was applied directly to Eq. (8) and not to Eq.

(11), one was left with another non-trivial decimation function for positive $b$. This hindered recognizing the role played by this equation for finite Jacobians. Because of the universal scaling properties of the reduced circle map $h$, $e_n(x_0)$ approaches a universal limit which can be obtained in terms of the universal fixed point of the circle map renormalization without solving $e(x)$ for arbitrary $x$ [9]. However, one has to be careful in the analysis because effectively it means calculating the limit of $\xi_{F_n}/b^{F_n}$ taking both the numerator and the denominator to zero (and $n$ to infinity).

In the case of the Harper equation, the continuous variable for $e_n$ is obtained from the fractional part $\{i\sigma\}$ of $i\sigma$, $x = (-\sigma)^n\{i\sigma\}$, and the resulting fixed point equation

$$e^*(x) = -e^*(\sigma^2 x + \sigma)e^*(-\sigma x) \qquad (17)$$

is solved by standard expansion methods [11]. We show this universal function in Fig. 1 which illustrates the nontriviality of the strong coupling fixed point as compared to the trivial weak coupling fixed point [8,10] of the Harper equation. As seen in the plot, the function is smooth and finite *almost* everywhere.

It is interesting that the above fixed point equation (17) was independently found by Kuznetsov et al. [14] in studying the birth of a strange nonchaotic attractor in a quasiperiodically forced map. In the next section, by using a discrete version of the Prüfer transformation, we relate the Harper equation directly to such a map.

## IV. LOCALIZATION AND STRANGE NONCHAOTIC ATTRACTORS

The Prüfer substitution [16] that was used to map the continuous Schrödinger equation to overdamped, quasiperiodically driven oscillator has motivated us to consider a similar transformation also in the discrete case. The main idea underlying the continuous transformation is to define new variables $\rho$ and $\alpha$ in terms of the wave function $\psi = \rho\cos(\alpha)$ and its derivative $\psi' = \rho\sin(\alpha)$ so that one obtains a first order equation for $\alpha$ where $\rho$ does not appear and where the potential of the quantum problem appears as a driving term. An analogous idea in the case of the discrete Harper equation would be to set e.g. $\psi_{i-1} = \rho_i\cos(\alpha_i)$, $\psi_i = \rho_i\sin(\alpha_i)$. In fact, the Harper equation gives trivially a very simple equation for $x_i = \cot(\alpha_i) = \psi_{i-1}/\psi_i$:

$$x_{i+1} = \frac{-1}{x_i - E + 2\lambda\cos[2\pi(i\sigma + \phi)]}. \qquad (18)$$

It is clear that higher iterations of this map generate continuous fraction expansions in terms of subsequent potential terms $V(i) = \lambda\cos[2\pi(i\sigma + \phi)]$. In the following, we study this one-dimensional map and verify some of the conjectures of Bondeson et al. [15] in the discrete context. Although the transformation above is extremely simple, it provides a useful tool to study various eigenvalue problems.

Numerical study of the map (18) shows that the extended phase of the Harper equation corresponds to invariant curves for the map. There are infinity of such curves depending upon the initial condition. The transition to localization in the Harper equation corresponds to transition from invariant curves to an attractor. Fig. 2 shows the numerically obtained Lyapunov exponent of the map as a function of the parameter $E$. As seen from the figures, the eigenenergies are singled out in this plot by giving rise to Lyapunov exponents larger than those of forbidden values of $E$. Furthermore, the Lyapunov exponent varies non-smoothly as the parameter $E$ is taken from allowed and to forbidden values. This in principal, could provide a new method to calculate the allowed energy spectrum of the eigenvalue problem.

As seen in Fig. 2, below criticality, the allowed energies give rise to quasiperiodic orbits (invariant curves) with zero Lyapunov exponent while the states in the forbidden regime are characterized by negative Lyapunov exponents. In the superctical regime, the Lyapunov exponent is always negative. At criticality, where the spectrum is conjectured to be singular continuous [4], the plot clearly illustrates the fact that the allowed energies form a Cantor set of zero measure. Another interesting aspect of the driven map is that the inverse localization length of the Harper equation gives the absolute value of the Lyapunov exponent. Since the localization length of the system is exactly known, this is an example of a system whose Lyapunov exponent is known exactly.

Finally, comparing Figs. 2 (a) and (b), we see a very interesting manifestation of the self-duality [4] of the Harper equation which can be written as

$$\gamma(\lambda, E) = -ln(\lambda) + \gamma(1/\lambda, E/\lambda). \qquad (19)$$

Fig. 3 shows the attractor of the mapping (18) at and slightly away from criticality. The self-similarity of the Harper wave function at critically implies that the attractor is fractal with structure at all scales. Interesting aspect of this attractor is the fact that it has a zero Lyapunov exponent. Therefore, the map exhibits a new type of SNA where the two nearby trajectories converge on the attractor, not exponentially but with a power law. In the supercritical regime, the attractor is associated with negative Lyapunov exponent which appears to smooth out the fragmented structure of the attractor. However, as we will argue later in this section, the attractor is in fact strange throughout the supercritical regime.

Another intriguing result emerging from this mapping is that in the localized regime, the critical phase $\phi_c$ of the potential leading to localization around the site $i = 0$ can be found as a "homoclinic" point where the forward and backward attractors of the map intersect (see Fig. 4).

It is interesting to note that independently of the value of $\lambda > 1$, a homoclinic point is always found at $\phi_c = 1/4$. In the strong coupling limit, the homoclinic points coincide with the divergences seen in the attractor. This point of view brings out the significance of the critical phase factor in the Harper equation which has not been understood in the past. Unlike the supercritical regime, at the critical point, the forward and backward attractors seem to overlay.

In the localized regime, the attractor is strange but nonchaotic. The appearance of the fractal structure can be easily seen coming down from the strong coupling limit $\lambda \to \infty$. Here we give a rather crude analysis of this problem showing just the main ideas of the argument.

Let us first rewrite the map in terms of the variables $x$ and $\theta$ assuming $E = 0$:

$$x' = \frac{-1}{x + 2\lambda \cos(2\pi\theta)} \tag{20}$$

$$\theta' = \theta + \sigma \;\; mod \;\; 1. \tag{21}$$

Note that the map remains invariant under the transformation $x \to -x$, $\theta \to \theta + 1/2$ so we can expect the attractor to have this symmetry as well. Therefore, in the following we assume implicitly that $\theta$ is considered mod $1/2$.

To the first order in $1/\lambda$, the attractor, whose invariant measure is always uniform in $\theta$, can be written as

$$x(\theta) = \frac{-1}{2\lambda \cos[2\pi(\theta - \sigma)]}. \tag{22}$$

However, the above form suggests a singularity appearing asymptotically as $\lambda \to \infty$ at $\theta_1 = 1/4 + \sigma$. This first order singularity gives rise to higher order singularities which are absent in the above approximation for the attractor. In order to see the appearance of the second order singularity, note that according to Eq. (22), $|x(\theta)|$ takes values in the interval $[1/(2\lambda), \infty)$ when $\theta$ is varied in $[0, 1/2)$. Thus, it is possible to find $\theta_2^*$ such that $x(\theta_2^* - \sigma) + 2\lambda \cos(2\pi\theta_2^*) = 0$ which gives rise to a singularity around $\theta_2 = \theta_2^* + \sigma$. Moreover, asymptotically $\theta_2^* \to \theta_1 + \sigma$ because the singularity around $\theta_1$ becomes very sharp and the attractor is mostly (as far as the invariant measure is concerned) close to zero away from $\theta_1$ as $\lambda \to \infty$. In other words, we have generated a new singularity close to $\theta_2 = 1/4 + 2\sigma$. Looking at the explicit (asymptotic) equation for $\theta_2^*$,

$$4\lambda^2 \cos(2\pi\theta_2^*) \cos[2\pi(\theta_2^* - \sigma)] = 1, \tag{23}$$

shows that the second order singularity is even sharper than the first order one. Now the same argument can be repeated to generate a third order singularity from the second order one and so on. Each new singularity is sharper than the previous one. Because the measure is uniform in $\theta$, the high order singularities become invisible for finite number of iterations of the map. Asymptotically with $\lambda \to \infty$, the singularities generated in this way appear at $\theta_m = 1/4 + m\sigma$ $(m = 1, 2, ...)$. For finite $\lambda$, their locations move a little bit but we still expect them to be dense in $\theta$. Their existence causes the attractor to be nowhere differentiable and to have the characteristics of a SNA [12].

In summary, the above reasoning provides a very strong argument for the existence of SNA in quasiperiodically driven maps given by Eq. (18).

## V. DISCUSSION

In this paper, we have shown an interesting relationship between the supercritical Harper equation and the critical dissipative standard map: the fluctuations in the localized eigenstates of the Harper equation are related to the tangent orbit of the standard map where the parameter $1/\lambda$ plays the role of the dissipation parameter $b$. In particular, the strong coupling limit of the Harper equation is analogous to the strong dissipation (or circle map) limit of the standard map. Inspite of these similarities, the two problems are quantitatively different due to the fact that they are characterized by different universal numbers $\zeta$. The roots for these two quantitatively different universality classes is tied to the fact that the dynamics governing the nonlinear potential in the Harper equation is a pure rotation while the dynamics underlying the critical dissipative standard map is highly nontrivial.

The same kind of reasoning can be applied also to another related class of problems studied by us recently [18], namely the phonon modes of the Frenkel-Kontorova model where the locations of the particles are described by the area-preseving standard map. Although this problem bears a close resemblance to the Harper equation, the supercritical regime in the Frenkel-Kontorova model has very different characteristics. Below criticality, the potential is derived from the dynamics analytically conjugate to the pure rotation in both models. However, beyond criticality, the Harper potential is still given by the pure rotation whereas the phonon potential is obtained from cantorus dynamics which cannot be mapped smoothly to the rotation. In the Harper equation, the "supercritical" regime is described by exponentially localized eigenfunctions with universal self-similar fluctuations characterized by a unique strong coupling fixed point [11]. In contrast, the phonon eigenmodes defy localization and remain critical with scaling characterized by a line of renormalization limit cycles. In addition, there exist an infinite sequence of parameter values in the supercritical regime where the renormalization limit cycle degenerates into a trivial fixed point. A very intriguing characteristics of these parameter values is the

fact that the corresponding phonon eigenfunction is represented by an infinite series of step-functions.

Our studies suggest that the solutions of eigenvalue problems where the potential is determined by sequences with complicated dynamics can be extremely rich. By studying the Harper equation and the phonons in the Frenkel-Kontorova model, we may have barely scratched the surface of the wealth of novel phenomena exhibited by these types of systems. For example, in the generalized versions of the Harper equations one encounters fat critical regimes with ergodic renormalization dynamics in the strong coupling limit [10,11].

The idea of relating an almost periodic eigenvalue problem to a quasiperiodically driven map has opened new possibilities for studying SNA. In this paper, we have shown that the existence of localization in the eigenvalue problem is associated with the appearance of homoclinic points in the corresponding map. Furthermore, the critical state at the onset to localization is found to correspond to a new type of strange attractor with zero Lyapunov exponent. In future, we also hope to answer the question about the class of models whose whole supercritical phase is described by a unique strong coupling limit of the theory. In particular, it would be interesting to know whether quasiperiodically driven maps studied earlier exhibit this type of universality or not.

The mapping from linear-difference equations to driven maps is of continuing interest to us. In future, we would like to apply this technique to a variety of other problems that can be written in the linear-difference form. In particular, the one to two-hole transition in the extended standard map [19], which corresponds to nonexponentially localized phonons [18], may provide new dynamics in the forced map. Fishman et al. [20] have shown that the quantum dynamics of the kicked rotor can be mapped to the tight binding model of the above form. The integrable rotor with quasiperiodic potential has been shown to exhibit Harper-type universality. However, according to our preliminary study on the nonintegrable rotor with pseudorandom potential, the localized phase is not characterized by a unique strong coupling fixed point in this model.

The research of IIS is supported by a grant from National Science Foundation DMR 093296. JAK would like to thank the organizers of the conference for their kind invitation.


[1] e-mail: isatija@sitar.gmu.edu.

[2] R. S. MacKay, Ph.D thesis, Princeton University (1982).

[3] T. Bohr, P. Bak and M. H. Jensen, Phys. Rev. A 30 (1984) 1970.

[4] B. Simon, Adv. Appl. Math. 3 (1982) 463; Y. Last, in Proceedings of the $XI^{th}$ international congress of mathematical physics, Ed. D. Iagolnitzer, International Press Inc., 1995, p. 366; S. Ya. Jitomirskaya, ibid. p. 373.

[5] P. G. Harper, Proc. Phys. Soc. London A 68 (1955) 874.

[6] M. J. Feigenbaum, L. P. Kadanoff and S. J. Shenker, Physica D 5 (1982) 370.

[7] S. Ostlund, D. Rand, J. Sethna and E. Siggia, Physica D 8 (1983) 303; D. A. Rand, Nonlinearity 5 (1992) 639, 663 and 681.

[8] S. Ostlund and R. Pandit, Phys. Rev. B 29 (1984) 1394; S. Ostlund, R. Pandit, D. Rand, H.J. Schellnhuber, and E. D. Siggia, Phys. Rev. Lett. 50 (1983) 1873.

[9] J. A. Ketoja, Phys. Rev. Lett. 69 (1992) 2180.

[10] J. A. Ketoja and I. I. Satija, Phys. Lett. A 194 (1994) 64; Physica A 219 (1995) 212; Phys. Rev. B 52 (1995) 3026.

[11] J. A. Ketoja and I. I. Satija, Phys. Rev. Lett. 75 (1995) 2762.

[12] C. Grebogi, E. Ott, S. Pelikan and J. A. Yorke, Physica D 13 (1984) 261.

[13] M. Ding. C. Grebogi and E. Ott, Phys. Rev. A 39 (1989) 2593; J. F. Heagy and S. M. Hammel, Physica D 70 (1994) 140. A. Pikovsky and U. Feudel, Chaos 5 (1995) 253; U. Feudel, J. Kurths and A. S. Pikovsky, Physica D 88 (1995) 176; O. Sosnovtseva, U. Feudel, J. Kurths and A. Pikovsky, Phys. Lett. A 218 (1996) 255; Y.-C. Lai, Phys. Rev. E 53 (1996) 57.

[14] S. P. Kuznetsov, A. S. Pikovsky and U. Feudel, Phys. Rev. E 51 (1995) R1629.

[15] A. Bondeson, E. Ott and T. M. Antonsen, Phys. Rev. Lett. 55 (1985) 2103.

[16] H. Prüfer, Math. Ann. 95 (1926) 499; B. Mielnik and M. A. Reyes, J. Phys. A 29 (1996) 6009.

[17] S. Aubry and G. André, Ann. Israel Phys. Soc. 3 (1980) 133.

[18] J. A. Ketoja and I. I. Satija, "Critical" phonons of the supercritical Frenkel-Kontorova model, to appear in Physica D.

[19] C. Baesens and R. S. MacKay, Physica D 71 (1994) 372.

[20] S. Fishman, D. R. Grempel and R. E. Prange, Phys. Rev. Lett. 49 (1982) 509; D. R. Grempel, R. E. Prange and S. Fishman, Phys. Rev. A 29 (1984) 1639.


FIG. 1. The renormalization fixed point for the fluctuations of the supercritical Harper equation obtained by solving the expansion of $1/e(x)$ upto the order $x^{23}$ by the Newton method. The fact that there has to be a singularity (or zero point for $1/e$) can be seen easily from the fixed point equation. The horizontal line shows the value of the universal $\zeta_H$.

FIG. 2. Lyapunov exponent $\gamma$ vs the parameter E for the one-dimensional map derived from the Harper equation. The figure obtained with $\lambda = 0.5$ (a) is equivalent to the figure with $\lambda = 2$ (b) after simple scaling of $\gamma$ and $E$ which illustrates the self-duality of the Harper equation. Fig. (c) shows the corresponding results at $\lambda = 1$. The allowed values of the energy are characterized by an attractor with zero Lyapunov exponent.

FIG. 3. The attractor of the one-dimensional map ($E = 0$) at the critical point $\lambda = 1$ (a) and at $\lambda = 1.05$ (b).

FIG. 4. The forward (lighter dots) and backward (darker dots) attractors of the map and the existence of homoclinic points for $\lambda = 2$. As $\lambda$ approaches $\infty$, the divergences coincide with the homoclinic points.

Fig1 Ketoja and Satija

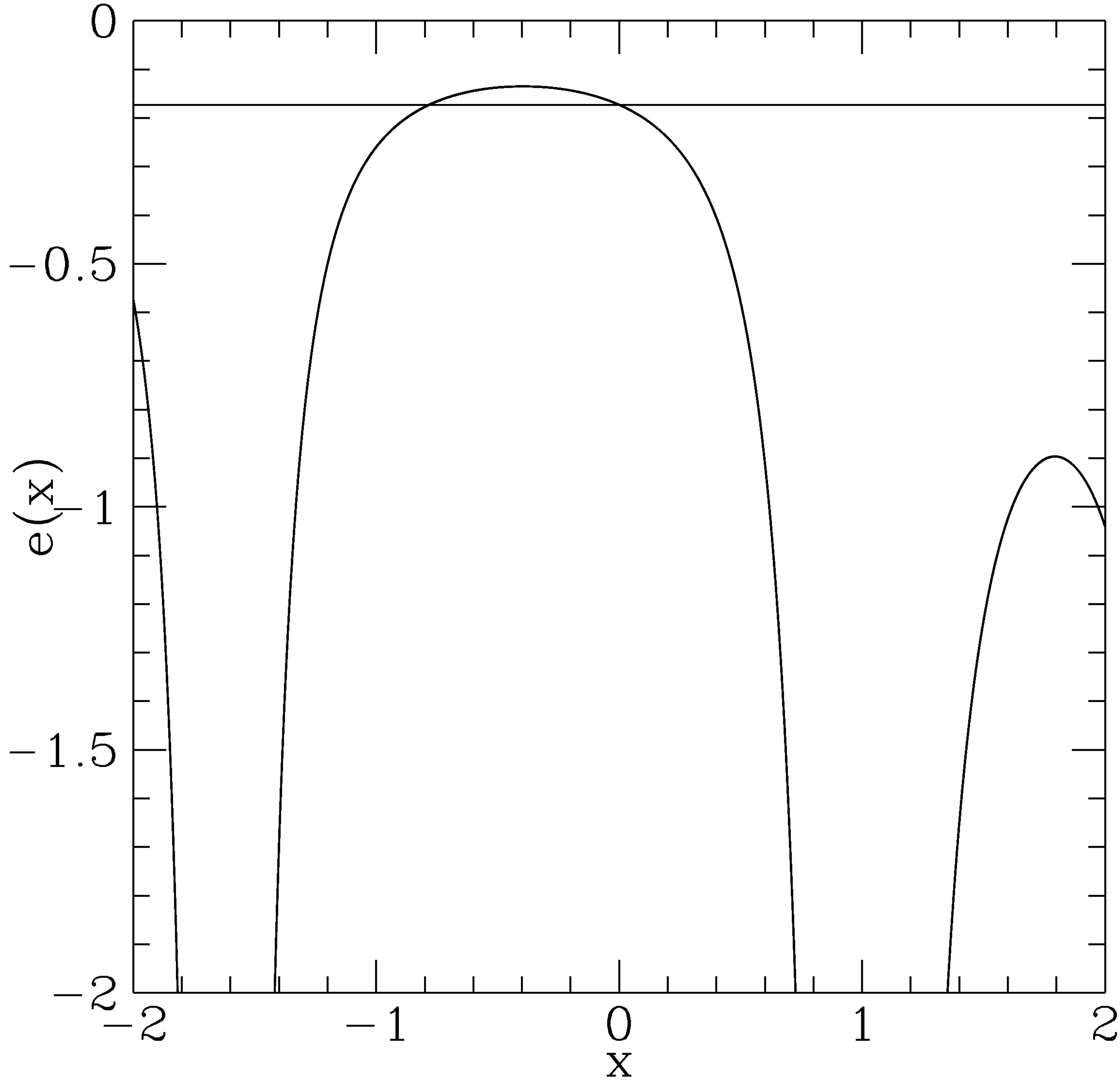
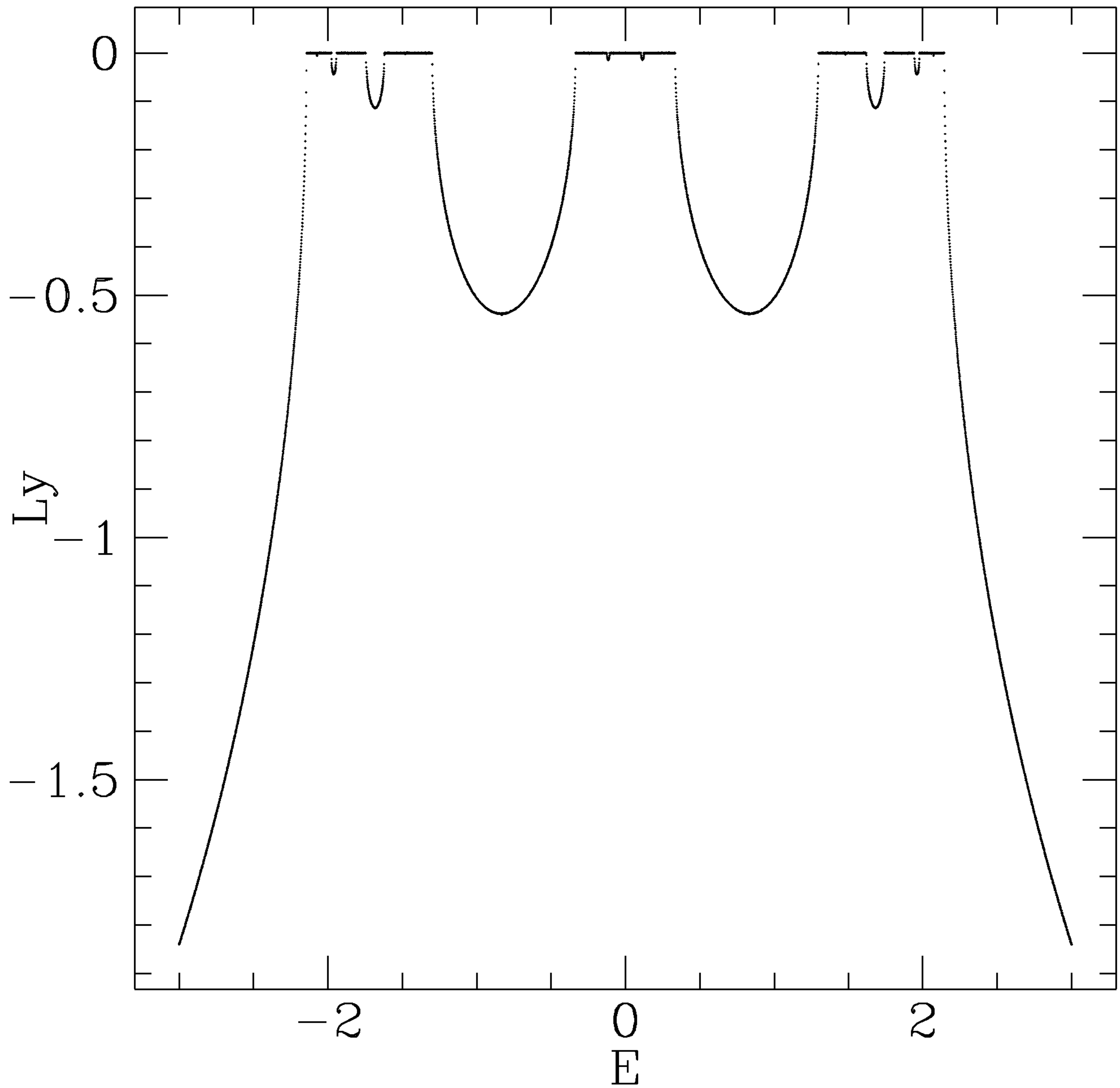


Fig2(a) Ketoja and Satija



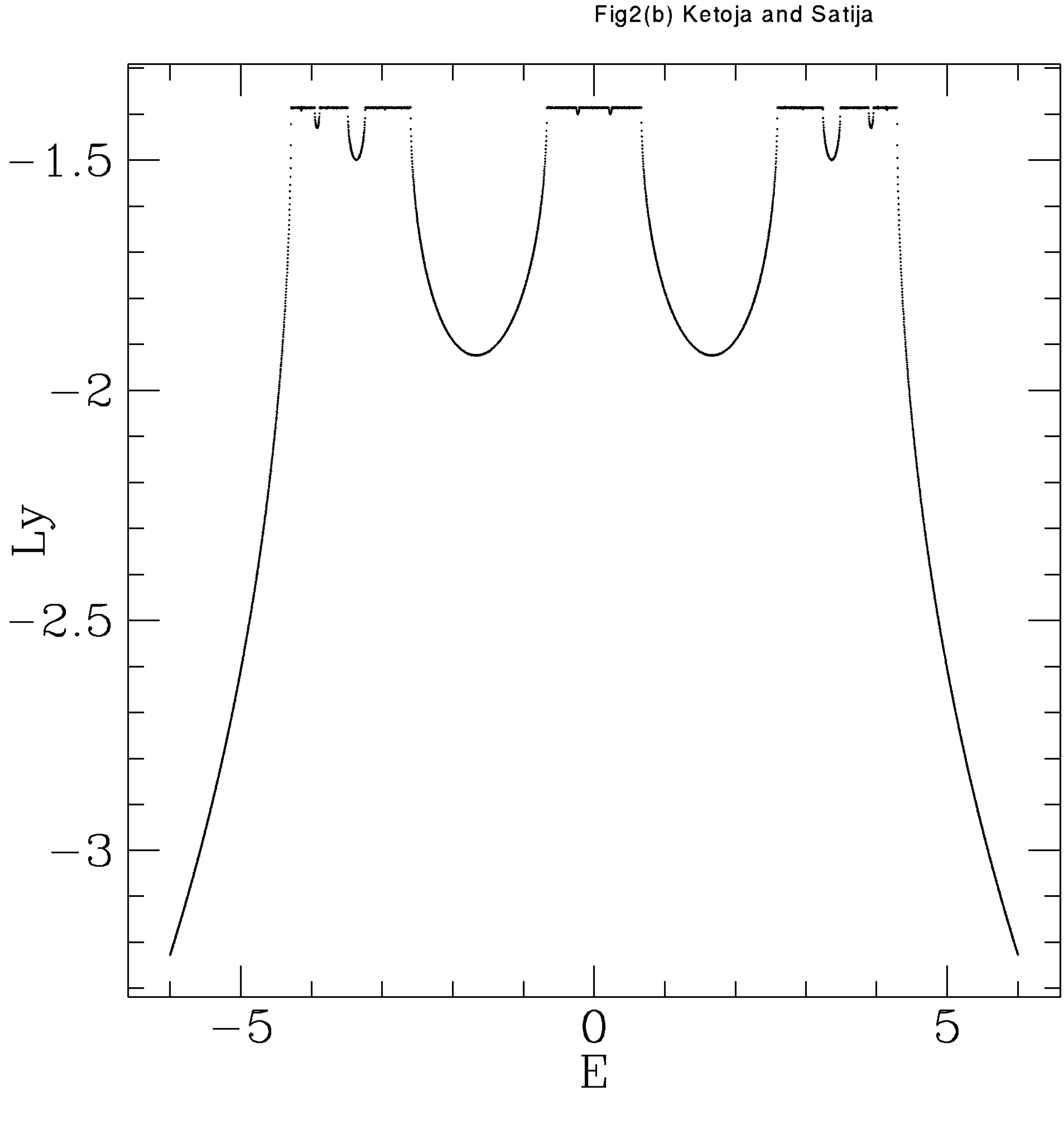


Fig2(b) Ketoja and Satija
−1.5
−2
−2.5
−3
Ly
−5
0
5
E

Fig2(c) Ketoja and Satija

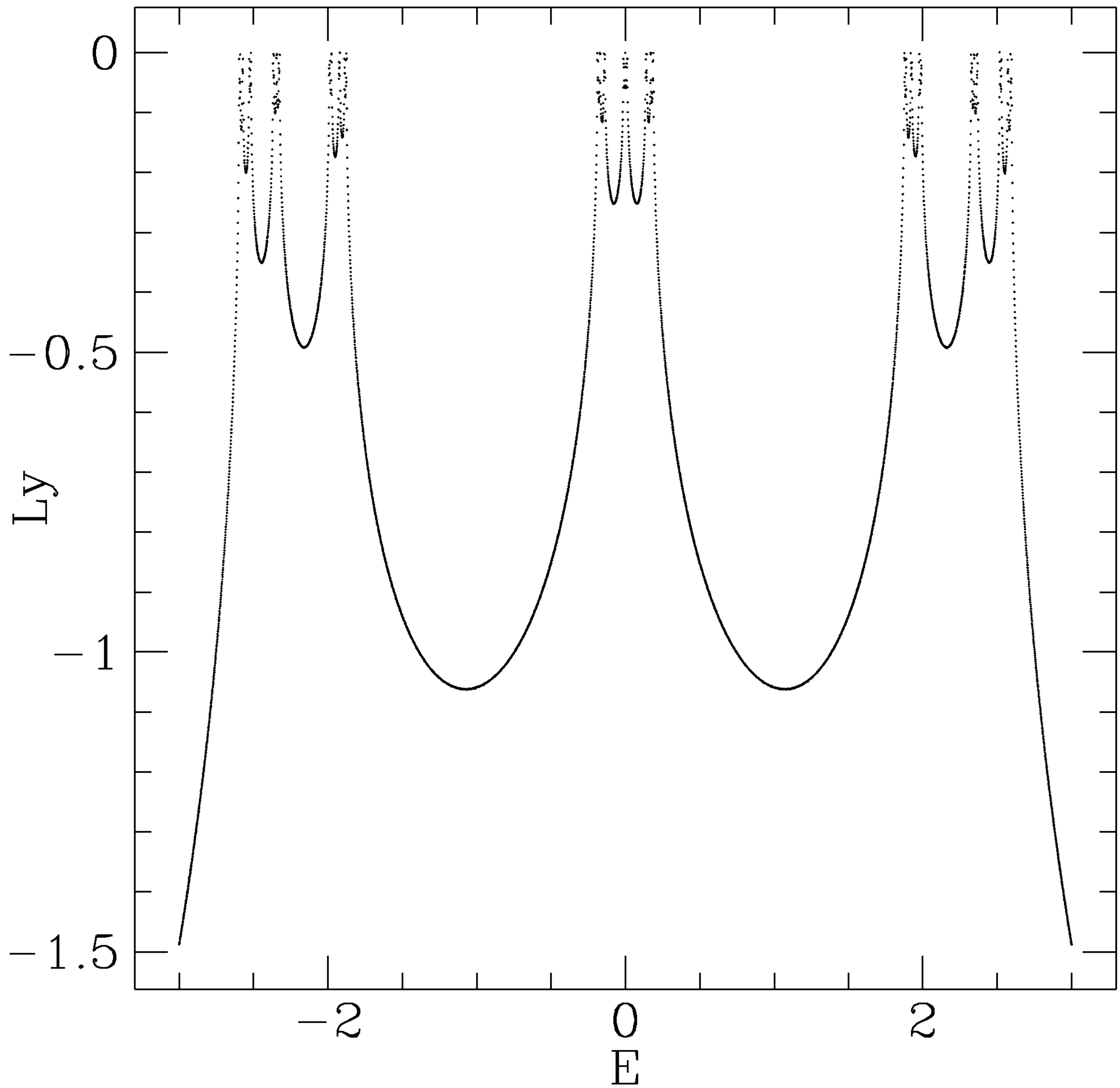


Fig3(a) Ketoja and Satija

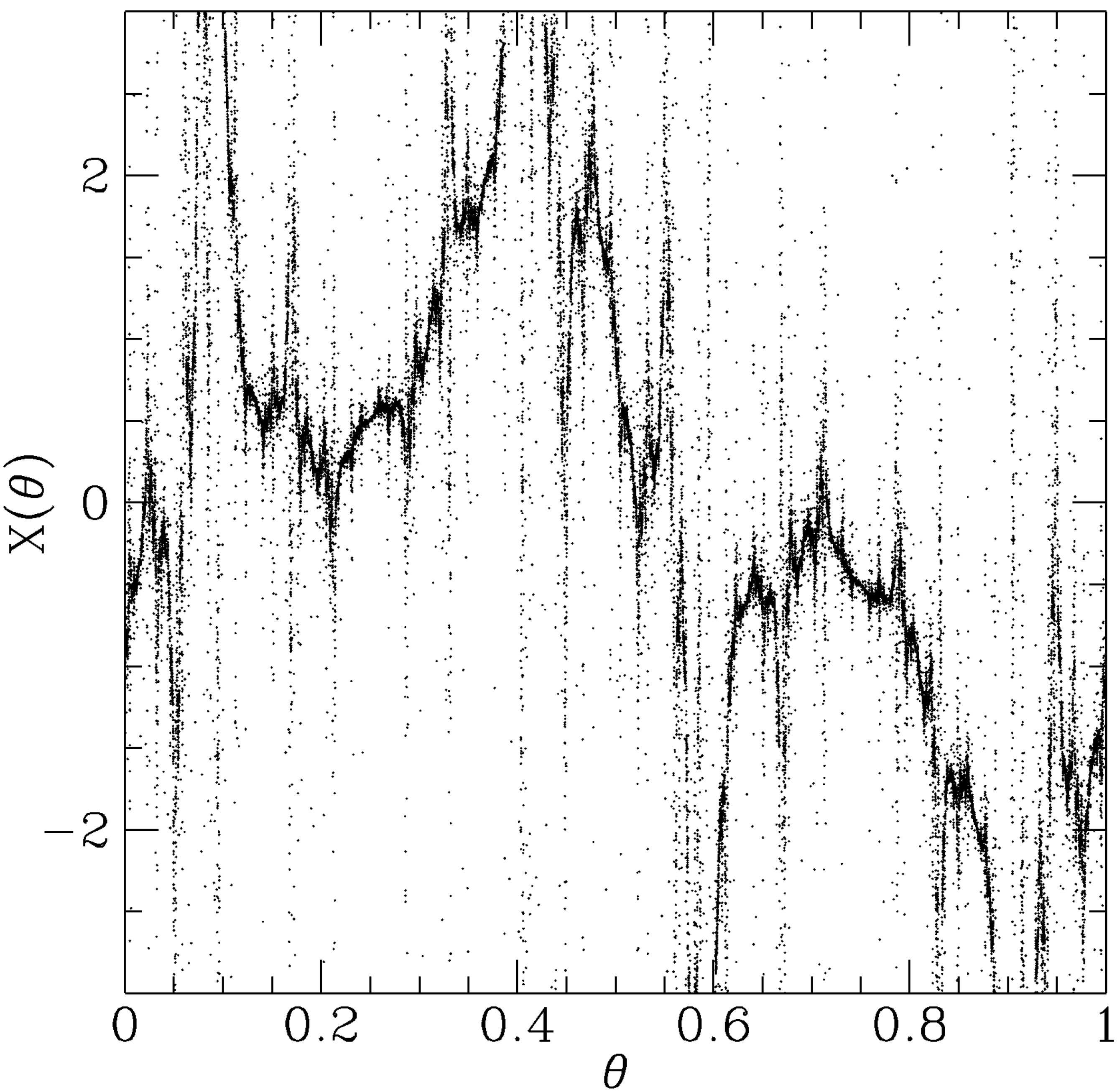


Fig3(b) Ketoja and Satija

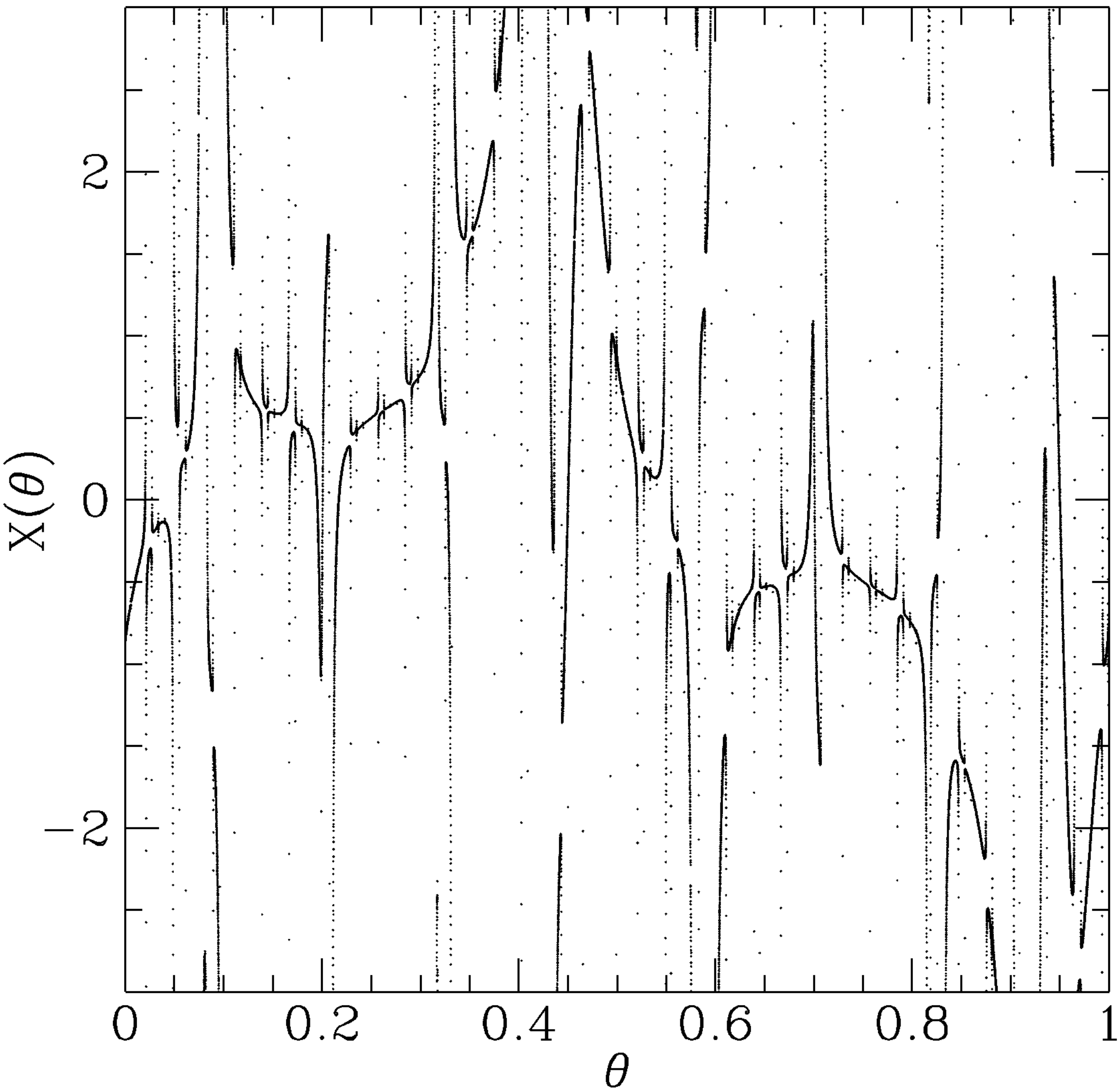


Fig4 Ketoja and Satija